$\mathcal{K}_g$ is not finitely generated

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1 Introduction

Let Σ_g be a closed orientable surface of genus g. The *mapping class group* Mod_g of Σ_g is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$. Recall that an essential simple closed curve γ in Σ_g is called a *bounding curve*, or *separating curve*, if it is null-homologous in Σ_g or, equivalently, if γ separates Σ_g into two connected components.

Let $\mathcal{K}_g$ denote the subgroup of Mod_g generated by the (infinite) collection of Dehn twists about bounding curves in Σ_g. Note that $\mathcal{K}_1$ is trivial. It has been a long-standing problem in the combinatorial topology of surfaces to determine whether or not the group $\mathcal{K}_g$ is finitely generated for $g \geq 2$. For a discussion of this problem, see, e.g., [Jo1, Jo3, Bi, Mo1, Mo3, Ak].

McCullough-Miller [MM] proved that $\mathcal{K}_2$ is not finitely generated; Mess then proved that $\mathcal{K}_2$ is in fact an infinite rank free group. Akita proved in [Ak] that for all $g \geq 2$, the rational homology $H_*(\mathcal{K}_g; \mathbb{Q})$ is infinite-dimensional as a vector space over $\mathbb{Q}$. Note that since $\mathcal{K}_g$ admits a free action on the Teichmuller space of Σ_g, which is contractible and finite-dimensional, $\mathcal{K}_g$ has finite cohomological dimension.

For some time it was not known if $\mathcal{K}_g$ was equal to, or perhaps a finite index subgroup of, the *Torelli group* $\mathcal{I}_g$, which is the subgroup of elements

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of Mod_g which act trivially on $H_1(\Sigma_g;\mathbb{Z})$. Powell [Po] proved that $\mathcal{K}_2 = \mathcal{I}_2$. Johnson proved in [Jo2] that for $g \geq 3$, the group $\mathcal{K}_g$ has infinite index in $\mathcal{I}_g$; he did this by constructing what is now called the *Johnson homomorphism*, which is the quotient map in the short exact sequence

$$1 \longrightarrow \mathcal{K}_g \longrightarrow \mathcal{I}_g \longrightarrow \wedge^3 H/H \longrightarrow 1$$

where $H = H_1(\Sigma_g;\mathbb{Z})$. Johnson then proved in [Jo3] that $\mathcal{I}_g$ is finitely generated for all $g \geq 3$. Our main result is the following.

Theorem 1.1. *The group $\mathcal{K}_g$ is not finitely generated for any $g \geq 2$.*

We will also prove along the way that that the once-punctured analogue of $\mathcal{K}_g$ is not finitely generated.

Theorem 1.1 answers Problem 10 of [Mo2], Problem 2.2(i) of [Mo3], and the question/conjecture on page 24 of [Bi]. We would still, however, like to know the answer to the following question, asked by Morita (see [Mo3], Problem 2.2(ii)).

Question 1.2. *Is $H_1(\mathcal{K}_g;\mathbb{Z})$ finitely generated for $g \geq 3$?*

Note that Birman-Craggs-Johnson (see, e.g. [BC, Jo1]) and Morita [Mo4] have found large abelian quotients of $\mathcal{K}_g$. We would also like to remark that Morita has discovered (see, e.g., [Mo4, Mo2, Mo3]) a strong connection between the algebraic structure of $\mathcal{K}_g$ and the Casson invariant for homology 3-spheres. For example, Morita proved in [Mo4] that every integral homology 3-sphere can be obtained by gluing two handlebodies along their boundaries via a map in $\mathcal{K}_g$; further, he has been able to express the Casson invariant as a homomorphism $\mathcal{K}_g \longrightarrow \mathbb{Z}$ (see, e.g., [Mo1]).

Rough outline of the proof. Our proof owes a great intellectual debt to the paper [MM] by D. McCullough and A. Miller, where the theorem is demonstrated in the genus 2 case; indeed we follow the same outline as their proof.

First, we find an action of $\mathcal{K}_g$ on the first homology of an abelian cover Y of Σ_g with Galois group $\mathbb{Z}^{2g-2}$. While $H_1(Y;\mathbb{Z})$ is infinitely generated, it is finitely generated as a module over the group-ring of the Galois group of the cover. We view this group-ring as the ring $\mathcal{L}_g$ of integral Laurent series

in $2g-2$ variables. This action a priori gives a rather complicated high-dimensional representation of $\mathcal{K}_g$. We first project to a Laurent series ring $\mathcal{L}$ in just one variable, and then are able to find and quotient out a codimension two fixed submodule. This reduction to a 2-dimensional representation is crucial for what follows. We then analyze this representation

$$\rho : \mathcal{K}_g \longrightarrow \mathrm{SL}_2(\mathcal{L})$$

The ring $\mathcal{L}$ comes equipped with a discrete valuation, and so $\mathrm{SL}_2(\mathcal{L})$ can be realized via Bruhat-Tits theory as a group of automorphisms of a certain simplicial tree. The Bass-Serre theory of graphs of groups—equivalently, of groups acting on trees—is especially suited to understanding whether or not such a group is finitely generated; one such criterion is proven in [MM]. To complete the proof, we compute enough about the image of ρ to apply this criterion to show that $\mathcal{K}_g$ is not finitely generated.

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2 Representing $\mathcal{K}_g$ on an abelian cover

Consider a standard symplectic basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ for $H_1(\Sigma_g; \mathbb{Z})$, where $a_i \cdot b_j = \delta_{i,j}$ and $a_i \cdot a_j = b_i \cdot b_j = 0$. Here and throughout this article, the symbol $\cdot$ is used to denote the algebraic intersection number of simple closed curves (or homology classes). By abuse of notation, we will also sometimes view the a_i and b_i as elements of $\pi_1(\Sigma_g)$, considered as relative to a fixed basepoint.

2.1 The abelian cover

Consider the free abelian group $\mathbb{Z}^{2g-2}$ with generators $\{s_2, \ldots, s_g, t_2, \ldots, t_g\}$ and the surjection $\psi : H_1(\Sigma_g; \mathbb{Z}) \to \mathbb{Z}^{2g-2}$ defined by

$$\begin{aligned} \psi(a_1) &= \psi(b_1) = 0 \\ \psi(a_i) &= s_i,\ i \geq 2 \\ \psi(b_i) &= t_i,\ i \geq 2 \end{aligned}$$

Composing with the Hurewicz map $\pi_1(\Sigma_g) \to H_1(\Sigma_g;\mathbb{Z})$ gives a surjection $\varphi : \pi_1(\Sigma_g) \to \mathbb{Z}^{2g-2}$; we denote the kernel of φ by K.

Let $p : Y \to \Sigma_g$ denote the covering corresponding to the subgroup $K \subset \pi_1(\Sigma_g)$. The group $\mathbb{Z}^{2g-2}$ then acts on Y by deck transformations. This action induces an action of $\mathbb{Z}^{2g-2}$ on $H_1(Y;\mathbb{Z})$, which is consequently a $\mathbb{Z}\left[s_2^{\pm 1}, \ldots, s_g^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}\right]$-module. We denote this Laurent series ring by $\mathcal{L}_g$.

It is rather easy to construct the cover Y explicitly. To this end, consider the decomposition of Σ_g into two subsurfaces $\Sigma_{g-1,1}$ and $\Sigma_{1,1}$ of genus $g-1$ and 1, respectively, obtained by cutting along the bounding curve representing the homotopy class $[a_1, b_1]$. Note that the subspace $H_1(\Sigma_{1,1};\mathbb{Z}) \subset H_1(\Sigma_g;\mathbb{Z})$ is the span of $\{a_1, b_1\}$. Let Y' denote the universal abelian cover of $\Sigma_{g-1,1}$, that is, the cover corresponding to the commutator subgroup of $\pi_1(\Sigma_{g-1,1})$. Since the boundary of $\Sigma_{g-1,1}$ is null-homologous, it lifts to a collection of simple closed boundary curves in Y', indexed by the set $\mathbb{Z}^{2g-2}$. We then obtain Y by gluing $\Sigma_{1,1}$ to each of these curves along its boundary.

The $\mathcal{L}_g$-module structure of $H_1(Y;\mathbb{Z})$ can now be read off from this geometric description of Y.

Proposition 2.1. *The homology group $H_1(Y;\mathbb{Z})$ is generated as an $\mathcal{L}_g$-module by the following $\binom{2g-2}{2} + 1$ elements: $a_1, b_1, [a_i, a_j]$ and $[b_i, b_j]$ for $2 \le i < j \le g$, and $[a_i, b_j]$ for $2 \le i, j \le g$, with $[a_g, b_g]$ excepted. Denote by W the submodule of $H_1(Y;\mathbb{Z})$ obtained by omitting the generators a_1 and b_1 from this list. We then have $p_*(a_1) = a_1, p_*(b_1) = b_1$, and $p_*(c) = 0$ for any $c \in W$. Moreover,*

$$\frac{H_1(Y;\mathbb{Z})}{W}$$

is a free $\mathcal{L}_g$-module on $\{a_1, b_1\}$.

Proof. It is a standard fact that the homology of the surface obtained by sewing in discs along the boundary circles of Y' is generated as an $\mathcal{L}_g$-module by the elements $[a_i, a_j]$, $[b_i, b_j]$, and $[a_i, b_j]$ (for the sake of normalization, we choose a single connected fundamental domain X for the action of K on Y and demand that all these generators be supported in X). Note that the resulting space is just the universal abelian cover of Σ_{g-1}. The element $[a_g, b_g]$ is omitted because the relation $[a_2, b_2] \cdot \cdots \cdot [a_g, b_g]$ in $\pi_1(\Sigma_{g-1})$ implies

that, in the homology of the cover, $[a_g, b_g]$ is in the span of the $[a_i, b_i]$ for $2 \leq i \leq g-1$.

The identification of the images of the generators under p_* follows directly from their definition. Finally, to compute $H_1(Y;\mathbb{Z})/W$, notice that W is the image of the natural map $H_1(Y';\mathbb{Z}) \to H_1(Y;\mathbb{Z})$ and that the map $H_1(Y, Y';\mathbb{Z}) \to H_0(Y';\mathbb{Z})$ is zero; then observe that Y/Y' is a wedge of tori, one for each element of $\mathbb{Z}^{2g-2}$. $\diamond$

We will need to compute the algebraic intersection numbers of certain curves in Y. To ease the exposition of the next result, it will be convenient to introduce another piece of notation. We denote the set $\{a_2, \ldots, a_g, b_2, \ldots, b_g\}$ by $\{c_1, \ldots, c_{2g-2}\}$ and the set $\{s_2, \ldots, s_g, t_2, \ldots, t_g\}$ by $\{u_1, \ldots, u_{2g-2}\}$. Thus, $H_1(Y)$ is generated as an $\mathcal{L}_g$-module by the elements a_1, b_1, and $[c_i, c_j]$ for $1 \leq i < j \leq 2g-2$ except $i = g$ and $j = 2g$.

Proposition 2.2. *Suppose $i, j, i', j' \in \{1, \ldots, 2g-2\}$ with $i \neq j$ and $i' \neq j'$. Assume first that $\{i, j\} \cap \{i', j'\} = \emptyset$. Then there exists $\epsilon_{i,j,i',j'} \in \{1, 0, -1\}$ such that*

$$[c_i, c_j] \cdot \left(u_1^{r_1} \cdots u_{2g-2}^{r_{2g-2}} [c_{i'}, c_{j'}]\right) =$$

$$= \begin{cases} (-1)^{r_i + r_j + r_{i'} + r_{j'}} \epsilon_{i,j,i',j'} & r_k \in \{\delta_{i,k}, \delta_{j,k}, -\delta_{i',k}, -\delta_{j',k}\} \text{ for all } k \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Now, assume that $i = i'$. Then there exists $\epsilon_{i,j,j'} \in \{1, 0, -1\}$ such that

$$[c_i, c_j] \cdot \left(u_1^{r_1} \cdots u_{2g-2}^{r_{2g-2}} [c_i, c_{j'}]\right) =$$

$$= \begin{cases} (-1)^{r_i + r_j + r_{j'}} \epsilon_{i,j,j'} & r_k \in \{\delta_{i,k}, \delta_{j,k}, -\delta_{i,k}, -\delta_{j',k}\} \text{ for all } k \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Lastly,

$$[c_i, c_j] \cdot \left(u_1^{r_1} \cdots u_{2g-2}^{r_{2g-2}} a_1\right) = [c_i, c_j] \cdot \left(u_1^{r_1} \cdots u_{2g-2}^{r_{2g-2}} b_1\right) = 0 \tag{3}$$

regardless of the integers r_k.

Proof. Equation (3) is clear since the curves in question are disjoint. To prove equation (1), notice that the curve representing the cycle $[c_i, c_j]$ is a kind of quadrilateral beginning at some basepoint y in the fundamental domain X, then passing to $u_i y$, followed by $u_i u_j y$, then $u_j y$, and then back to the original basepoint y. The curve $[c_{i'}, c_{j'}]$ thus intersects $[c_i, c_j]$ only once, at y, but this intersection is not necessarily transverse, so we cannot determine the value of $\epsilon_{i,j,i',j'} = [c_i, c_j] \cdot [c_{i'}, c_{j'}]$ aside from observing that it lies in the set $\{1, 0, -1\}$.

Now, the curve $u_1^{r_1} \cdots u_{2g-2}^{r_{2g-2}} [c_{i'}, c_{j'}]$ cannot possibly meet $[c_i, c_j]$ unless $r_k \in \{\delta_{i,k}, \delta_{j,k}, -\delta_{i',k}, -\delta_{j',k}\}$ for all k. On the other hand, if the two curves do meet, then by symmetry, their intersection numbers are determined by $[c_i, c_j] \cdot [c_{i'}, c_{j'}]$, as indicated in the statement of the proposition.

The verification of equation (2) proceeds in much the same way. The only subtlety comes in checking the cases $r_i = 0$, in which the curves in question actually have an entire segment in common. But one can perturb one of the curves so that they only meet at one endpoint of the segment; the computation then follows from the usual symmetry. $\diamond$

2.2 The representation

It will be useful for us to consider pointed versions of $\mathcal{I}_g$ and $\mathcal{K}_g$. We work with respect to the basepoint $x = p(y) \in \Sigma_g$. Denote by $\mathcal{I}_{g,*}$ the group of components of the group of basepoint-preserving diffeomorphisms of Σ_g which act trivially on $H_1(\Sigma_g; \mathbb{Z})$. "Forgetting the basepoint" clearly gives a surjective homomorphism $\mathcal{I}_{g,*} \to \mathcal{I}_g$. Denote by $\mathcal{K}_{g,*}$ the subgroup of $\mathcal{I}_{g,*}$ generated by twists about bounding curves which avoid the basepoint. Again, the operation of forgetting the basepoint induces a surjection $\mathcal{K}_{g,*} \to \mathcal{K}_g$.

Recall that $K = \pi_1(Y)$. Note that since K is not a characteristic subgroup of $\pi_1(\Sigma_g)$, an arbitrary mapping class need not lift to Y. In fact, there are even elements of $\mathcal{I}_g$ which don't lift to Y. However, we have the following.

Proposition 2.3. *Each element of $\mathcal{K}_{g,*}$ has a lift to a basepoint-preserving diffeomorphism of Y which is unique up to basepoint-preserving isotopy.*

Proof. The uniqueness is clear. Moreover, by the universal lifting property for covering maps, the collection of basepoint-preserving mapping classes

that admit such a lift constitutes a subgroup. Thus, we need only verify the result for Dehn twists about bounding curves, as these generate $\mathcal{K}_{g,*}$.

To this end, let C be a bounding curve on Σ_g, and denote by t_C the twist about C. Since p is an abelian cover, C lifts to a simple closed curve in Y. Consider the map $\widetilde{t_C}$, which is a simultaneous Dehn twist about all the lifts of C. This obviously constitutes a lift of t_C. $\diamond$

These observations are enough to give us our main tool. Henceforth C will denote an arbitrary bounding curve in Σ_g, and $\widetilde{C}$ will denote a lift of C to Y. The homology class of $\widetilde{C}$ will be written

$$c + \sum m_{p_2,\ldots,p_g,q_2,\ldots,q_g} s_2^{p_2} \cdots s_g^{p_g} t_2^{q_2} \cdots t_g^{q_g} a_1 + n_{p_2,\ldots,p_g,q_2,\ldots q_g} s_2^{p_2} \cdots s_g^{p_g} t_2^{q_2} \cdots t_g^{q_g} b_1$$

where $c \in W$ (recall W was defined in the statement of Proposition 2.1), the sum is taken over all integers $p_2, \ldots, p_g, q_2, \ldots, q_g$, and the m's and n's are integral coefficients, all but finitely many of which vanish. To simplify the notation, we will use underlined symbols to refer to $(g-1)$-tuples of objects indexed by the set $\{2, \ldots, g\}$. For example, $\underline{p}$ will stand for $p_2, \ldots, p_g$, the symbol $\underline{s}$ will stand for $s_2, \ldots, s_g$ and, crucially, binary operations on underlined quantities will be performed componentwise, so that $\underline{s}^{\underline{p}} = s_2^{p_2} \cdots s_g^{p_g}$.

We are now ready to lift the action of $\mathcal{K}_{g,*}$.

Proposition 2.4. *The operation which associates to an element of $\mathcal{K}_{g,*}$, the action of its lift to Y on $H_1(Y;\mathbb{Z})$ gives rise to a representation*

$$\widetilde{\rho} : \mathcal{K}_{g,*} \to \mathrm{Aut}_{\mathcal{L}_g}(H_1(Y;\mathbb{Z}))$$

Proof. We must check that $\widetilde{\rho}$ takes composition to multiplication and that its image respects the $\mathcal{L}_g$-action on $H_1(Y;\mathbb{Z})$. The former condition follows from the uniqueness up to isotopy of lifts; the latter holds because for any bounding curve C in Σ_g, the set of all lifts of C to Y is $\mathcal{L}_g$-invariant. $\diamond$

2.3 Reducing dimension

The representation $\widetilde{\rho}$ is quite complicated, because $H_1(Y;\mathbb{Z})$ is a rather large module. We instead would like to work with a 2-dimensional $\mathcal{L}_g$-representation. We will achieve this by proving that $\widetilde{\rho}$ contains a large subrepresentation, namely W, that we will be able to ignore. In order to do this we first need to analyze the image under $\widetilde{\rho}$ of a twist about a bounding curve.

Proposition 2.5. *Let C be a bounding curve on Σ_g and $1 \leq i < j \leq 2g-2$. Then*

$$\tilde{\rho}(t_C)([c_i, c_j]) = [c_i, c_j] + d$$

where d can be written as a sum of terms each of which is divisible by $(u_k - 1)(u_l - 1)$ for some $1 \leq k < l \leq 2g-2$.

Proof. We first assume that $\tilde{C} = [c_{i'}, c_{j'}]$. Recall that if $\beta = \{\beta_k\}$ is a family of mutually disjoint and nonisotopic simple closed curves on a surface, and if α is another simple closed curve, then the homology class of the twist $t_\beta(\alpha)$ of α about β is

$$[t_\beta(\alpha)] = [\alpha] + \sum_k (\alpha \cdot \beta_k)[\beta_k] \tag{4}$$

Now, if $\{i, j\} = \{i', j'\}$, then of course $\tilde{\rho}(t_C)([c_i, c_j]) = [c_i, c_j]$. If, instead, $\{i, j\} \cap \{i', j'\} = \emptyset$, then equation (1) tells us that

$$\tilde{\rho}(t_C)([c_i, c_j]) = [c_i, c_j] + \epsilon_{i,j,i',j'}(u_i - 1)(u_j - 1)(u_{i'}^{-1} - 1)(u_{j'}^{-1} - 1)[c_{i'}, c_{j'}]$$

which is of the desired form if we set $k = i$ and $l = j$. Lastly, suppose that $\{i, j\} \cap \{i', j'\}$ contains a single element, say without loss of generality $i = i'$. Then equation (2) gives us

$$\begin{aligned} \tilde{\rho}(t_C)([c_i, c_j]) &= [c_i, c_j] + \epsilon_{i,j,,j'}\left(-u_i^{-1} + 1 - u_i\right)(u_j - 1)\left(u_{j'}^{-1} - 1\right)[c_{i'}, c_{j'}] \\ &= [c_i, c_j] + \epsilon_{i,j,,j'}u_{j'}^{-1}\left(u_i^{-1} - 1 + u_i\right)(u_j - 1)(u_{j'} - 1)[c_{i'}, c_{j'}] \end{aligned}$$

which again gives us what we want, with $k = j$ and $l = j'$.

The general case follows from this calculation by the linearity present in equation (4) along with the vanishing of equation (3). ◇

We now use Proposition 2.5 to find a substantially smaller representation of $\mathcal{K}_{g,*}$. Denote by $\mathcal{L}$ the Laurent series ring $\mathbb{Z}[t, t^{-1}]$, and define $\Phi : \mathcal{L}_g \longrightarrow \mathcal{L}$ by $\Phi(s_i) = 1$ for $2 \leq i \leq g$ and

$$\Phi(t_i) = \begin{cases} t & \text{if } i = 2 \\ 1 & \text{if } 3 \leq i \leq g \end{cases}$$

The homomorphism Φ induces a homomorphism

$$\hat{\Phi} : \mathrm{Aut}_{\mathcal{L}_g}(M) \longrightarrow \mathrm{Aut}_{\mathcal{L}}(M \otimes_{\mathcal{L}_g} \mathcal{L})$$

for any $\mathcal{L}_g$-module M. Now define

$$\hat{\rho} : \mathcal{K}_{g,*} \to \mathrm{Aut}_{\mathcal{L}} \left(H_1(Y;\mathbb{Z}) \otimes_{\mathcal{L}_g} \mathcal{L} \right)$$

by

$$\hat{\rho} = \hat{\Phi} \circ \widetilde{\rho}$$

Recall now that $W \subset H_1(Y;\mathbb{Z})$ was defined in the statement of Proposition 2.1.

Corollary 2.6. *The representation $\hat{\rho}$ becomes trivial when restricted to $W \otimes_{\mathcal{L}_g} \mathcal{L}$.*

Proof. Proposition 2.5 guarantees that for any bounding curve C and for any $1 \le i < j \le 2g-2$, that

$$\widetilde{\rho}(t_C)([c_i, c_j]) = [c_i, c_j] + d$$

where d is a sum of terms each of which is divisible by $(u_k - 1)(u_l - 1)$ for some $1 \le k < l \le 2g-2$. Since $k \neq l$, at least one of u_k and u_l is not equal to t_2, so it must be the case that each of the summands of d vanishes when we tensor with $\mathcal{L}$. Thus, $\hat{\rho}(t_C)([c_i, c_j]) = [c_i, c_j]$. The desired result then follows from the fact that the t_C generate $\mathcal{K}_{g,*}$ and the $[c_i, c_j]$ generate $W \otimes_{\mathcal{L}_g} \mathcal{L}$. $\diamond$

We are now able to define the representation that will actually allow us to prove our result. Since the image of $\hat{\rho} : \mathcal{K}_{g,*} \to \mathrm{Aut}_{\mathcal{L}} \left(H_1(Y;\mathbb{Z}) \otimes_{\mathcal{L}_g} \mathcal{L} \right)$ fixes $W \otimes_{\mathcal{L}_g} \mathcal{L}$, we may pass to a quotient representation

$$\check{\rho} : \mathcal{K}_{g,*} \longrightarrow \mathrm{Aut}_{\mathcal{L}} \left[\frac{H_1(Y;\mathbb{Z}) \otimes_{\mathcal{L}_g} \mathcal{L}}{W \otimes_{\mathcal{L}_g} \mathcal{L}} \right] \approx \mathrm{GL}_2(\mathcal{L})$$

where the last isomorphism follows from Proposition 2.1.

The first thing we will need to know about $\check{\rho}$ is the following.

Proposition 2.7. *The image of $\check{\rho}$ is actually contained in $\mathrm{SL}_2(\mathcal{L})$ rather than $\mathrm{GL}_2(\mathcal{L})$. Moreover, for a bounding curve C on Σ_g, we have*

$$\check{\rho}(t_C) = \Phi \begin{pmatrix} 1 + \sum n_{\underline{i},\underline{j}} m_{\underline{p},\underline{q}} \underline{s}^{\underline{p}-\underline{i}} \underline{t}^{\underline{q}-\underline{j}} & -\sum m_{\underline{i},\underline{j}} m_{\underline{p},\underline{q}} \underline{s}^{\underline{p}-\underline{i}} \underline{t}^{\underline{q}-\underline{j}} \\ \sum n_{\underline{i},\underline{j}} n_{\underline{p},\underline{q}} \underline{s}^{\underline{p}-\underline{i}} \underline{t}^{\underline{q}-\underline{j}} & 1 - \sum m_{\underline{i},\underline{j}} n_{\underline{p},\underline{q}} \underline{s}^{\underline{p}-\underline{i}} \underline{t}^{\underline{q}-\underline{j}} \end{pmatrix} \tag{5}$$

Furthermore, $\check{\rho}$ descends to a representation

$$\rho : \mathcal{K}_g \longrightarrow \mathrm{SL}_2(\mathcal{L})$$

Proof. Observe that the statement that the image of $\check{\rho}$ lies in $\mathrm{SL}_2(\mathcal{L})$ rather than $\mathrm{GL}_2(\mathcal{L})$ follows formally from equation (5), so it suffices to verify that equality. To establish that, we compute before projecting to $\mathcal{L}$ via Φ by simply expanding out the summations

$$(\widetilde{t}_C)_*(a_1) \equiv a_1 + \sum (a_1 \cdot \underline{s}^{\underline{i}}\underline{t}^{\underline{j}}\widetilde{C})\underline{s}^{\underline{i}}\underline{t}^{\underline{j}}[\widetilde{C}] \pmod W$$

and

$$(\widetilde{t}_C)_*(b_1) \equiv b_1 + \sum (b_1 \cdot \underline{s}^{\underline{i}}\underline{t}^{\underline{j}}\widetilde{C})\underline{s}^{\underline{i}}\underline{t}^{\underline{j}}[\widetilde{C}] \pmod W$$

using the formulas

$$a_1 \cdot \underline{s}^{\underline{i}}\underline{t}^{\underline{j}}\widetilde{C} = n_{-\underline{i},-\underline{j}}$$

and

$$b_1 \cdot \underline{s}^{\underline{i}}\underline{t}^{\underline{j}}\widetilde{C} = -m_{-\underline{i},-\underline{j}}$$

To verify the last statement, consider an element η of $\mathcal{K}_{g,*}$ that lies in the kernel of the projection $\mathcal{K}_{g,*} \to \mathcal{K}_g$. Denote by $\widetilde{\eta}$ the basepoint-preserving lift of η to Y. Since η is isotopic to the identity once we forget basepoints, $\widetilde{\eta}$ must be isotopic to a diffeomorphism covering the identity map on Σ_g. Thus, we must have an equation

$$\check{\rho}(\eta) = \begin{pmatrix} \Phi\left(\underline{s}^{\underline{p}}\underline{t}^{\underline{q}}\right) & 0 \\ 0 & \Phi\left(\underline{s}^{\underline{p}}\underline{t}^{\underline{q}}\right) \end{pmatrix}$$

But in order for this to lie in SL_2, it must be the identity matrix, so $\check{\rho}$ factors through the quotient $\mathcal{K}_g$ of $\mathcal{K}_{g,*}$. $\diamond$

3 Amalgamated products and infinite generation

Denote by H the image of the homomorphism $\rho : \mathcal{K}_g \longrightarrow \mathrm{SL}_2(\mathcal{L})$. Our goal is to prove that H is not finitely generated. We now describe how we will do this.

Consider the inclusion $\mathrm{SL}_2(\mathcal{L}) \subset \mathrm{SL}_2(\mathbb{Q}[t, t^{-1}])$. The field $\mathbb{Q}(t)$ obtained by adjoining a free variable t to the rational numbers is equipped with a discrete valuation and contains $\mathbb{Q}[t, t^{-1}]$, so one can apply the construction of Bruhat-Tits-Serre to find a (locally infinite) simplicial tree on which

$\mathrm{SL}_2(\mathbb{Q}[t,t^{-1}])$ acts by isometries. The Bass-Serre theory of groups acting on trees can then be applied (see [BM], §5) to express $\mathrm{SL}_2(\mathbb{Q}[t,t^{-1}])$ as an amalgamated product:

$$\mathrm{SL}_2(\mathbb{Q}[t,t^{-1}]) \cong A *_{\mathcal{U}} B \tag{6}$$

where $A = \mathrm{SL}_2(\mathbb{Q}[t])$,

$$B = \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

and $\mathcal{U} = A \cap B$.

This decomposition allows one to apply the theory of graphs of groups to obtain the following criterion, which is Proposition 5 in [MM].

Proposition 3.1 (Criterion for infinite generation). *Let $A *_{\mathcal{U}} B$ be an amalgamated product, and let H be any subgroup. Suppose there exist elements $M_k \in A \backslash \mathcal{U}$ and $N_k \in B \backslash \mathcal{U}$ such that*

1. *$M_k N_k M_k^{-1} \in H$; and*
2. *$(H \cap A) M_k \mathcal{U} \neq (H \cap A) M_l \mathcal{U}$ whenever $k \neq l$.*

Then H is not finitely generated.

We apply Proposition 3.1 to the situation above, with $\mathrm{SL}_2(\mathbb{Q}[t,t^{-1}]) \cong A *_{\mathcal{U}} B$ and with $H = \rho(\mathcal{K}_g)$. Our goal now is to find matrices M_k and N_k satisfying the desired criterion.

3.1 The elements M_k and N

For a positive integer k, we let

$$M_k = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathcal{L})$$

We also set

$$N = \begin{pmatrix} 1 & t-2+t^{-1} \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathcal{L})$$

We now verify that the first hypothesis of Proposition 3.1 holds in our case; here we are taking $N_k = N$ for all k.

Proposition 3.2. *For each $k \geq 1$, the matrix $M_k N M_k^{-1}$ lies in H.*

Proof. First of all, consider the simple closed bounding curve C shown in Figure 1. The figure is drawn so that the homology of the leftmost handle of Σ_g is spanned by $\{a_1, b_1\}$. We now lift C to Y; this is shown in Figure 2.

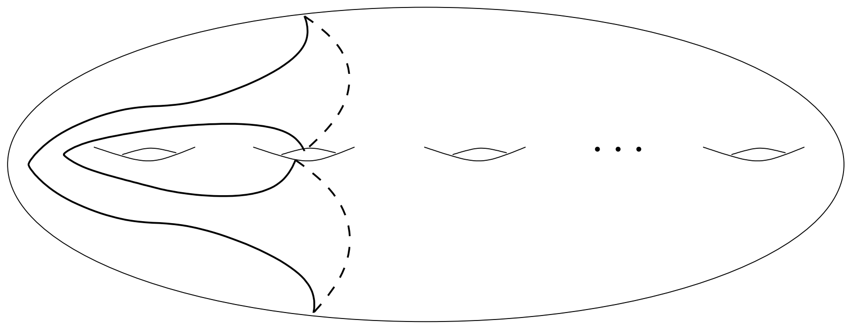

Figure 1: The curve C

Here, each octogan with a handle coming out of it corresponds to a single fundamental domain for the $\mathbb{Z}^{2g-2}$-action on Y; we have drawn the two lifts of C that meet the fundamental domain X. In general, of course, the base is a $4(g-1)$-gon; the figure corresponds to the case $g = 3$. It is clear that no lifts of C meet a_1, so that $\rho(t_C)(a_1) = a_1$. Moreover, by twisting b_1 about the two curves shown in Figure 2, one sees that

$$\rho(t_C)(b_1) = b_1 + \Phi(t_2 - 2 + t_2^{-1})a_1 = b_1 + (t - 2 + t^{-1})a_1$$

and therefore that $\rho(t_C) = N$.

Secondly, since b_1 is in the kernel of the map $\pi_1(\Sigma_g) \longrightarrow \mathbb{Z}^{2g-2}$, the twist t_{b_1} lifts to Y. Denoting by T the simultaneous twist about all the lifts of b_1 to Y, we see that $M_k = T_*^k$. Set $C' = t_{b_1}^k(C)$. We then have

$$\begin{aligned} M_k N M_k^{-1} &= T_*^k \left(\widetilde{t}_C\right)_* T_*^{-k} \\ &= \left(\widetilde{t}_{C'}\right)_* \end{aligned}$$

the last equality following from the general formula $f t_a f^{-1} = t_{f(a)}$, where f is any mapping class and t_a any Dehn twist. Since C' bounds in Σ_g, we see that $M_k N M_k^{-1} = \rho(t_{C'})$. $\diamond$

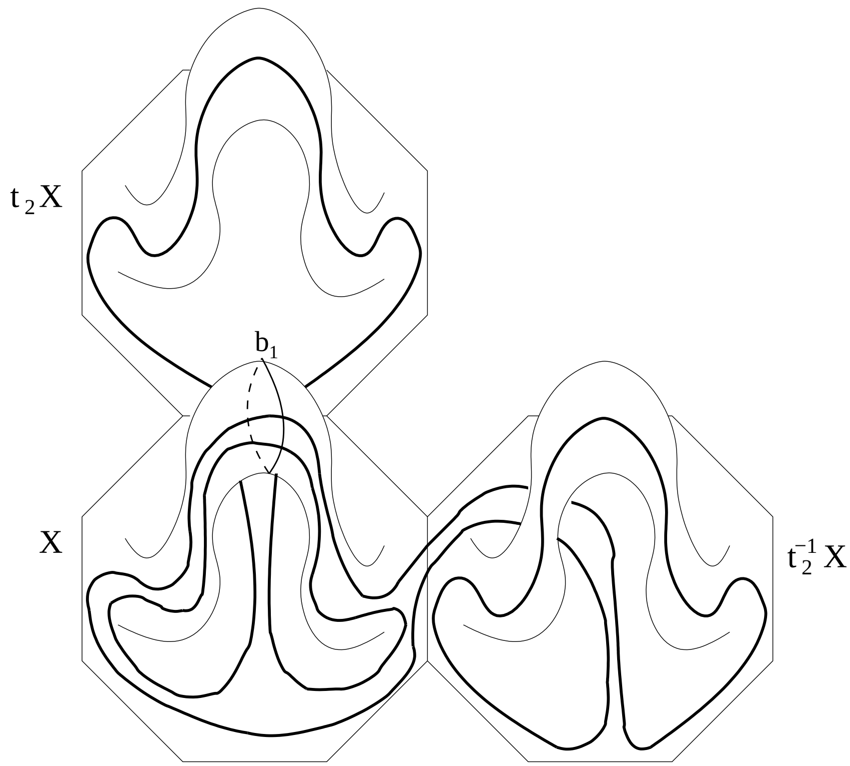

Figure 2: The two lifts of C that meet b_1.

3.2 Distinctness of double cosets

The rest of this paper is devoted to proving the following.

Proposition 3.3. *With the notation as above, we have*

$$(H \cap A)M_k\mathcal{U} \neq (H \cap A)M_l\mathcal{U}$$

for all $k \neq l$.

Given Proposition 3.3, whose proof we present in the next section, we are now able to establish our main result, Theorem 1.1.

Proof of Theorem 1.1. We apply Proposition 3.1 to the subgroup $H = \rho(\mathcal{K}_g)$ of $\mathrm{SL}_2(\mathbb{Q}[t,t^{-1}]) \cong A *_{\mathcal{U}} B$, with M_k and $N_k = N$ as above. First observe that $M_k \in A \backslash \mathcal{U}$ since t does not divide k, and $N \in B \backslash \mathcal{U}$ since $t - 2 + t^{-1} \notin \mathbb{Q}[t]$. Therefore, in light of Propositions 3.2 and 3.3, Proposition 3.1 implies that H is not finitely generated. As $\mathcal{K}_g$ surjects onto H, it is not finitely generated. $\diamond$

Note that since $\mathcal{K}_{g,*}$ surjects onto $\mathcal{K}_g$, it follows that $\mathcal{K}_{g,*}$ is also not finitely generated.

4 The proof of Proposition 3.3

In this section we prove Proposition 3.3. In order to do this we will prove that the elements of $H = \rho(\mathcal{K}_g)$ are of a very special form. To state this precisely, we will need the following.

Definition 4.1 (Balanced polynomials). Let $f \in \mathbb{Z}[u_1^{\pm 1}, \ldots, u_n^{\pm 1}]$ be a Laurent polynomial in n variables over the integers. We say that f is *balanced* if

1. $f(1, 1, \ldots, 1) = 0$; and
2. for all n-tuples $(i_1, \ldots, i_n) \in \mathbb{Z}^n$, the coefficients of $u_1^{i_1} \cdots u_n^{i_n}$ and $u_1^{-i_1} \cdots u_n^{-i_n}$ in f are equal.

Parallel with a crucial observation of McCullough-Miller [MM], we have the following.

Proposition 4.2. *Each element of H has the form*

$$\begin{pmatrix} 1+P_1 & Q_1 \\ Q_2 & 1-P_2 \end{pmatrix}$$

where $P_1, P_2, Q_1,$ and Q_2 are balanced.

Proof. Recall the map $\Phi : \mathcal{L}_g \to \mathcal{L}$ above. We begin by fixing an element $T = \rho(t_C) \in H$ and writing

$$T = \begin{pmatrix} \Phi(1+R_1) & \Phi(S_1) \\ \Phi(S_2) & \Phi(1-R_2) \end{pmatrix}$$

where now the R_i and S_i lie in $\mathcal{L}_g$. Equation (5) gives us expressions for $R_1, R_2, S_1,$ and S_2 in terms of the m and n coefficients. Since the twist t_C lies in the Torelli group $\mathcal{I}_g$, we have

$$\rho_*\left(\left(\widetilde{t}_C\right)_*\right)(a_1) = \rho_*(a_1) = a_1.$$

But equation (5) tells us that

$$\rho_* \left(\left(\widetilde{t}_C \right)_* \right) (a_1) = a_1 + R_1(1,1,\ldots,1)a_1 + S_2(1,1,\ldots,1)b_1$$

and so $R_1(1,1,\ldots,1) = S_2(1,1,\ldots,1) = 0$. A similar analysis of b_1 allows us to conclude that $R_2(1,1,\ldots,1) = S_1(1,1,\ldots,1) = 0$.

It follows via formal manipulations from equation (5) that S_1 and S_2 also satisfy the other criterion for balancedness. We now turn our attention to R_1 and R_2. Notice that for all $\underline{p}$ and $\underline{q}$, we have

$$\begin{aligned} 0 &= \widetilde{C} \cdot \underline{s}^{-\underline{p}} \underline{t}^{-\underline{q}} \\ &= \sum_{\underline{i},\underline{j}} m_{\underline{i},\underline{j}} n_{\underline{i}+\underline{p},\underline{j}+\underline{q}} - \sum_{\underline{i},\underline{j}} n_{\underline{i},\underline{j}} m_{\underline{i}+\underline{p},\underline{j}+\underline{q}} \end{aligned}$$

From this, it follows that $R_1 = R_2$, from which one can deduce formally that R_1 is balanced.

Since it is clear that Φ takes balanced polynomials to balanced polynomials, we have the desired property for elements of the form $\rho(t_C)$. But the set of elements of $\mathrm{SL}_2(\mathcal{L})$ of the desired form is evidently a subgroup, so the result follows since the t_C generate $\mathcal{K}_g$. $\diamond$

Following Lemma 7 in [MM], we will now see how Proposition 3.3 follows rather formally from Proposition 4.2.

Proof of Proposition 3.3. Suppose, that the M_k and M_l are in the same double coset, that is, that we have a matrix equation

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} = \begin{pmatrix} 1+P_1 & Q_1 \\ Q_2 & 1-P_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \begin{pmatrix} u & v \\ wt & z \end{pmatrix} \tag{7}$$

with

$$\begin{pmatrix} 1+P_1 & Q_1 \\ Q_2 & 1-P_2 \end{pmatrix} \in H \cap A$$

and

$$\begin{pmatrix} u & v \\ wt & z \end{pmatrix} \in \mathcal{U}$$

By Proposition 4.2, we know that P_1, P_2, Q_1, and Q_2 are balanced. By the definition of A, they also lie in $\mathbb{Q}[t]$. Thus, they are constant and hence

vanish. Therefore, setting $t = 0$ in equation (7) gives

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \begin{pmatrix} u(0) & v(0) \\ 0 & z(0) \end{pmatrix} = \begin{pmatrix} u(0) & v(0) \\ lu(0) & lv(0) + z(0) \end{pmatrix}$$

which obviously implies that $k = l$. $\diamond$

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